2016/17 MATH2230B/C Complex Variables with Applications
Suggested Solution of Selected Problems in HW 1
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P.61 8,9 will be graded

All the problems are from the textbook, Complex Variables and Application ($9th$ edition).

1 P.16

7. Show that
$$\left|\text{Re}(2+\bar{z}+z^3)\right| \le 4 \quad \text{when} \quad |z| \le 1.$$

Proof. Let z be such that $|z| \le 1$, then we can write z into polar form:
$$z = re^{i\theta},$$
where $0 \le r \le 1$ and $-\pi < \theta \le \pi$. Hence, we have
$$\begin{aligned}\left|\text{Re}(2+\bar{z}+z^3)\right| &= \left|\text{Re}(2+re^{-i\theta}+r^3e^{i3\theta})\right| \\ &= |2+r\cos\theta+r^3\cos 3\theta| \\ &\le 2+r|\cos\theta|+r^3|\cos 3\theta| \\ &\le 2+r+r^3 \\ &\le 2+1+1=4.\end{aligned}$$
□

13. Show that the equation $|z-z_0| = R$ of a circle, centered at z_0 with radius R, can be written as
$$|z|^2 - 2\text{Re}(z\bar{z}_0) + |z_0|^2 = R^2.$$

Proof. From $|z-z_0| = R$, we obtain
$$\begin{aligned} |z-z_0| &= R \\ \iff |z-z_0|^2 &= R^2 \\ \iff (z-z_0)(\bar{z}-\bar{z}_0) &= R^2 \\ \iff (z\bar{z}) - (z\bar{z}_0+\bar{z}z_0) + z_0\bar{z}_0 &= R^2 \\ \iff |z|^2 - 2\text{Re}(z\bar{z}_0) + |z_0|^2 &= R^2 \end{aligned}$$

Note that we have used the following facts:
$$|\omega|^2 = \omega\bar{\omega} \quad \forall \omega \in \mathbb{C},$$
$$\omega + \bar{\omega} = 2\text{Re}(\omega) \quad \forall \omega \in \mathbb{C}.$$
□

14. Show that the hyperbola $x^2 - y^2 = 1$ can be written as

$$z^2 + \bar{z}^2 = 2,$$

where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

Proof. Note that the following hold: for any $z \in \mathbb{C}$

$$x = \mathrm{Re}(z) = \frac{z + \bar{z}}{2},$$

$$y = \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}.$$

Then,

$$\begin{aligned} x^2 - y^2 &= 1 \\ \iff \left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2 &= 1 \\ \iff \frac{2(z^2+\bar{z}^2)}{4} &= 1 \\ \iff z^2 + \bar{z}^2 &= 2. \end{aligned}$$

□

2 P.23

1. Find the principal argument $Arg(z)$ when
   (a) $z = \frac{-2}{1+\sqrt{3}i}$;
   (b) $z = (\sqrt{3} - i)^6$.

Solution. *(a) Let* $z = \frac{-2}{1+\sqrt{3}i}$, *then*

$$\begin{aligned} z &= \frac{-2(1-\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)} \\ &= \frac{-2(1-\sqrt{3}i)}{4} \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i. \end{aligned}$$

The principal value of z, $\theta = Arg(z)$ *is obtained by*

$$\sin\theta = \frac{\sqrt{3}}{2} \quad \textit{and} \quad \cos\theta = -\frac{1}{2},$$

$$\theta = \frac{2}{3}\pi.$$

□

(b) *Let* $\omega = \sqrt{3} - i$*, then we write* ω *into polar form:*

$$\omega = 2e^{-i\frac{\pi}{6}}.$$

Hence, we have

$$z = \omega^6 = 2^6 e^{-i\pi} = -64.$$

Obviously, the principal argument of z *is* $Arg(z) = \pi$*. (Note that* $-\pi < Arg(z) \le \pi$ *for any* $z \in \mathbb{C}$*.)* □

3 P.31

3. Find $(-8 - 8\sqrt{3}i)^{1/4}$, express the roots in rectangular coordinates, exhibit then as the vertices of a certain square, and point out which is the principal root.

Solution. *Let* $z = -8 - 8\sqrt{3}i$*, then write it into polar form to obtain*

$$z = 16e^{-i2\pi/3}.$$

Then, we have

$$z^{1/4} = 2e^{i\left(-\frac{\pi}{6} + \frac{2k\pi}{4}\right)}, \quad k = 0, 1, 2, 3.$$

The principal root of z *is* $2e^{-i\pi/6} = \sqrt{3} - i$*.* □

6. Find the four zeros of the polynomial $z^4 + 4$, one of them being

$$z_0 = \sqrt{2}e^{i\pi/4} = 1 + i.$$

Then use those zeros to factor $z^4 + 4$ into quadratic factors with real coefficients.

Remark. Typo in the textbook: one should factorize $z^4 + 4$ into quadratic factors, not $z^2 + 4$.

Solution. *Since* z_0 *is the zero of* $z^4 + 4$*, then* $\bar{z}_0$ *is also the zero of* $z^4 + 4$*. On the other hand, it is easy to verify that* $-z_0$ *is the zero of* $z^4 + 4$ *as well. Then, so is* $-\bar{z}_0$*. Hence, we can factor the polynomial* $z^4 + 4$ *into the following*

$$z^4 + 4 = (z - z_0)(z - \bar{z}_0)(z + z_0)(z + \bar{z}_0).$$

Note that

$$(z - z_0)(z - \bar{z}_0) = z^2 - (z_0 + \bar{z}_0)z + |z_0|^2 = z^2 - 2z + 2,$$

$$(z + z_0)(z + \bar{z}_0) = z^2 + (z_0 + \bar{z}_0)z + |z_0|^2 = z^2 + 2z + 2.$$

Then, the polynomial $z^4 + 4$ *can be factored into quadratic factors with real coefficients*

$$z^4 + 4 = (z^2 - 2z + 2)(z^2 + 2z + 2).$$

□

4 P.61

8. Show that $f'(z)$ does not exist at any point z when

   (a) $f(z) = \text{Re}(z)$.

   (b) $f(z) = \text{Im}(z)$.

Proof. (a) Let $f(z) = \text{Re}(z)$, then for any $z \in \mathbb{C}$, as Δz approaches the origin horizontally through $(\Delta x, 0)$ on the real axis,

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\text{Re}(z+\Delta x) - \text{Re}(z)}{\Delta x} = \frac{\text{Re}(z) + \Delta x - \text{Re}(z)}{\Delta x} = 1.$$

On the other hand, as Δz approaches the origin vertically through $(0, \Delta y)$ on the imaginary axis,

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\text{Re}(z+i\Delta y) - \text{Re}(z)}{i\Delta y} = \frac{\text{Re}(z) - \text{Re}(z)}{i\Delta y} = 0.$$

It follows that $\frac{df}{dz}$ does not exist anywhere.

(b) Similar to part (a),

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\text{Im}(z+\Delta x) - \text{Im}(z)}{\Delta x} = \frac{\text{Im}(z) - \text{Im}(z)}{\Delta x} = 0$$

and

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\text{Im}(z+i\Delta y) - \text{Im}(z)}{i\Delta y} = \frac{\text{Im}(z) + \Delta y - \text{Im}(z)}{i\Delta y} = -i.$$

□

9. Let f denote the function whose values are

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz or $\Delta x \Delta y$ plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in the plane. Conclude from these observations that $f'(0)$ does not exist.

Proof. Let $z = 0$ and $\Delta z = \Delta x$ on the real axis, then

$$\frac{f(z+\Delta z)}{\Delta z} = \frac{f(\Delta z)}{\Delta z} = \frac{f(\Delta x)}{\Delta x} = \frac{(\Delta x)^2/\Delta x}{\Delta x} = 1.$$

Similarly, if we take $\Delta z = i\Delta y$ on the imaginary axis, then

$$\frac{f(z+\Delta z)}{\Delta z} = \frac{f(i\Delta y)}{i\Delta y} = \frac{(-i\Delta y)^2/i\Delta y}{i\Delta y} = 1.$$

On the other hand, if we take $\Delta z = \Delta x + i\Delta x = \Delta x(1+i)$, then

$$\begin{aligned} \frac{f(z+\Delta z)}{\Delta z} &= \frac{f(\Delta x(1+i))}{\Delta x(1+i)} \\ &= \frac{(\Delta x(1-i))^2/(\Delta x(1+i))}{\Delta x(1+i)} \\ &= \frac{(\Delta x(1-i))^2}{(\Delta x(1+i))^2} \\ &= \left(\frac{1-i}{1+i}\right)^2 = (-i)^2 = -1. \end{aligned}$$

If $f'(z)$ exists at $z = 0$, then the limit

$$\lim_{\Delta \to 0} \frac{f(z+\Delta z)}{\Delta z}$$

can be found by letting Δz approach the origin in the complex plane in any manner. **Hence,** $f'(0)$ **does not exist**. Remark that it is not sufficient to consider only horizontal and vertical approaches to the origin in the complex plane $\mathbb{C}$. □